MIDTERM DIFFERENTIAL EQUATIONS SUMMARY

(1) First Order
   (a) Linear
      - Homogeneous: $y' + p(t)y = 0$
      - Non-homogeneous(use variation of parameters): $y' + p(t)y = g(t)$.
   (b) Separable: $\frac{dy}{dt} = \frac{f(t)}{g(y)}$
   (c) Exact: $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ with $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} = 0$
   (d) (Separable) Homogeneous $\frac{dy}{dt} = f(y/t)$
   (e) General equation: $\frac{dy}{dt} = f(x,y)$
      - Picard iterates
      - Existence of solutions

(2) Second Order
   (a) Liner
      - Homogeneous: $y'' + p(t)y' + q(t)y = 0$
         - General theory: existence/uniqueness of solutions, linear independence of functions (Wronskian).
         - Constant coefficients: $ay'' + by' + cy = 0$
            * Repeated roots
            * Complex roots
         - Euler's equation: $at^2y'' + bty' + cy = 0$
            * Repeated roots
            * Complex roots
         - Series Solutions
            * Regular equation
            * Singular points
         - Reduction of order: given y_1, find y_2.
      - Non-homogeneous (variation of parameters): $y'' + p(t)y' + q(t)y = g(t)$

1. First Order Equations

1.1. Linear.

- Homogeneous: $y' + p(t)y = 0$
  - Rewrite as $\frac{y'}{y} = -p(t)$
  - Integrate both sides
$$\ln|y(t)| = -\int p(t)dt$$
  - Exponentiate both sides
$$y = Ce^{-\int p(t)dt}$$
    where C is arbitrary.
- Non-homogeneous equation: $y' + p(t)y = g(t)$ (Variation of parameters)
  - First solve the corresponding homogeneous equation $y_h' + p(t)y_h = 0 \Longrightarrow y_h = e^{-\int p(t)dt}$
  - Look for a solution of the form
$$y = u(t)y_h(t).$$
  - Plugging into the original equation and simplifying we have
$$u'(t)y_h(t) = g(t)$$
$$u'(t) = g(t)e^{\int p(t)dt}$$
$$y = e^{-\int p(t)dt}\int g(t)e^{\int p(t)dt}dt$$
    (Don't forget the constant of integration when evaluating $\int g(t)e^{\int p(t)dt}dt$)

1.2. Separable.

- $\frac{dy}{dt} = \frac{f(t)}{g(y)}$
- Rewrite as
$$g(y)\frac{dy}{dt} = f(t)$$
- Integrate both sides
$$\int g(y)\frac{dy}{dt}dt = \int f(t)dt$$
- Note that
$$\int g(y)\frac{dy}{dt}dt = \int g(y)dy$$
- Solve for y (if you can)
- Equation is called **homogeneous** (not to be confused with linear homogeneous) if it is given by
$$\frac{dy}{dt} = f(\frac{y}{t}).$$
  One can then introduce a change of variables $v = \frac{y}{t}$ to get
$$t\frac{dv}{dt} + v = f(v)$$
  which is a separable equation.

1.3. Exact.

- $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ with $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} = 0$
- Find $\psi(t,y)$ such that

$$\frac{\partial \psi}{\partial t} = M$$
$$\frac{\partial \psi}{\partial y} = N.$$

- The differential equation then becomes

$$\frac{d}{dt}\psi = M(t,y) + N(t,y)\frac{dy}{dt} = 0$$

and the solutions are given by

$$\psi(t,y) = \text{const.}$$

1.4. Existence of Solutions.

- $y' = f(y,t)$; $y(t_0) = y_0$
- Consider Picard iterates

$$\begin{aligned} y_1(t) &= y_0 + \int_{t_0}^{t} f(y_0, s)ds \\ y_2(t) &= y_1(t_0) + \int_{t_0}^{t} f(y_1(s), s)ds \\ &\vdots \\ y_k(t) &= y_{k-1}(t_0) + \int_{t_0}^{t} f(y_{k-1}(s), s)ds \end{aligned}$$

Theorem. *Let f and $\frac{\partial f}{\partial y}$ be continuous in the rectangle $R : t_0 \le t \le t_0 + a$, $|y - y_0| \le b$. Compute*

$$M = \max_{(t,y)\in R} |f(t,y)|$$

and set

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

Then the initial-value problem $y' = f(y,t)$; $y(t_0) = y_0$ has a unique solution $y(t)$ on the interval $t_0 \le t \le t_0 + \alpha$ given by

$$y(t) = \lim_{k\to\infty} y_k(t)$$

where y_k are the Picard iterates.

2. Second Order Linear Equations

2.1. General theory.

Theorem 1. Given differential equation

$$y'' + p(t)y' + q(t)y = 0$$

if $p(t)$ and $q(t)$ are continuous on an interval $\alpha < t < \beta$, then there exists a unique solution to the equation with prescribed initial values $y(t_0), y'(t_0)$ for any $t_0 \in (\alpha, \beta)$.

- The above theorem implies that the general solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$ is given by

$$y = c_1 y_1 + c_2 y_2$$

where y_1, y_2 are two linearly independent solutions.

- To see if functions y_1, y_2 are linearly independent, compute the **Wronskian**
$$W(y_1, y_2)(t) := y_1(t)y_2'(t) - y_2(t)y_1'(t)$$
If the Wronskian is not the zero function, then functions y_1, y_2 are linearly independent.
- **Reduction of order:** Suppose we know one solution $y_1(t)$. A linearly independent solution is given by
$$y_2 = u(t)y_1(t)$$
where $u(t)$ satisfies
$$\frac{du}{dt} = \frac{e^{-\int p(t)dt}}{y_1(t)^2}.$$
See pg. 146 of textbook for a proof.

2.2. Constant coefficients.

- $ay'' + by' + cy = 0$ where $a, b, c \in \mathbb{R}$ are constants.
- Use the ansatz $y = e^{rt}$
- Plugging into the original equation and simplifying we get the characteristic equation
$$ar^2 + br + c = 0.$$
- There are three cases
  - Characteristic equation has two distinct real roots r_1, r_2: linearly independent solutions are
$$y_1 = e^{r_1 t} \quad y_2 = e^{r_2 t}$$
  - Characteristic equation has a double real root r: linearly independent solutions are
$$y_1 = e^{rt} \quad y_2 = te^{rt}$$
  - Characteristic equation has complex roots $r = \alpha \pm \beta i$: linearly independent solutions are
$$y_1 = e^{\alpha t}\sin\beta t \quad y_2 = e^{\alpha t}\cos\beta t$$

2.3. Eulers equation.

- $at^2y'' + bty' + cy = 0$ on the interval $(0, \infty)$ (similar story for $(-\infty, 0)$)
- Use the ansatz $y = t^r$
- Plugging into the original equation and simplifying we get the characteristic equation
$$ar(r-1) + br + c = 0.$$
- There are three cases
  - Characteristic equation has two distinct real roots r_1, r_2: linearly independent solutions are
$$y_1 = t^{r_1} \quad y_2 = t^{r_2}$$
  - Characteristic equation has a double real root r: linearly independent solutions are
$$y_1 = t^r \quad y_2 = \ln(t)t^r$$
  - Characteristic equation has complex roots $r = \alpha \pm \beta i$: linearly independent solutions are
$$y_1 = t^\alpha \sin(\beta \ln(t)) \quad y_2 = t^\alpha \cos(\beta \ln(t))$$

2.4. **Series solution .**

- Consider an equation $y'' + p(t)y' + q(t)y = 0$ where $p(t), q(t)$ are analytic functions around a point t_0, i.e.
$$p(t) = \sum_{i=0}^{\infty} p_i(t - t_0)$$
$$q(t) = \sum_{i=0}^{\infty} q_i(t - t_0)$$
with radii of convergence R_p, R_q respectively. Then the solutions to the differential equation are analytic at t_0, i.e.
$$y(t) = \sum_{i=0}^{\infty} a_i(t - t_0)$$
with radius of convergence R_y satisfying $R_y \geq \min(R_p, R_q)$.
- To find the solutions using the series method, plug in the ansatz $y(t) = \sum_{i=0}^{\infty} a_i(t-t_0)$ into the differential equation.
- Consider an equation $P(t)y'' + Q(t)y' + R(t)y = 0$ where $P(t), Q(t), R(t)$ are analytic functions. We can rewrite this equation as $y'' + p(t)y' + q(t)y = 0$. A point t_0 is a **singular** point if $P(t)$ has a zero at t_0 and $p(t)$ or $q(t)$ cannot be extended as analytic functions at t_0. A singular point t_0 is called **regular** if
$$(t - t_0)p(t) = \sum_{i=0}^{\infty} p_i(t - t_0)$$
$$(t - t_0)^2 q(t) = \sum_{i=0}^{\infty} q_i(t - t_0)$$
i.e. if $(t-t_0)p(t)$ and $(t-t_0)^2 q(t)$ can be extended as analytic functions at t_0. The **indicial equation** is then
$$r(r - 1) + p_0 r + q_0 = 0.$$

2.5. **Non-homogeneous equation- Variation of Parameters .**

- Consider a non-homogeneous equation $y'' + p(t)y' + q(t)y = g(t)$. A general solution has the form
$$y = y_p + c_1 y_1 + c_2 y_2$$
where c_1, c_2 are arbitrary constants, y_p is any solution of the non-homogeneous equation and y_1, y_2 form a fundamental set of solutions of the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$.
- First solve the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ to find y_1, y_2. A particular solution is given by
$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$
where
$$\frac{du_1}{dt} = \frac{-g(t)y_2(t)}{W(y_1, y_2)(t)}$$
$$\frac{du_2}{dt} = \frac{g(t)y_1(t)}{W(y_1, y_2)(t)}$$
see pg. 153 of the textbook for a proof.